
Problem Tutorial: “Parallel Universes”

Let’s call the graph that needs to be made connected the first one, and the graph whose connectivity cannot be violated the second one.

Suppose the second graph contains an edge (v, u) such that the vertices v and u are in the same connected component of the first graph. Then consider any vertex a from a different component of the first graph. Notice that one of the operations (v, a) and (u, a) does not violate the connectivity of the second graph. Thus, it is possible to perform one operation to merge the component of vertex a with the component of vertices v and u , which solves the problem.

Now, notice that if there exists a pair of vertices v and u such that there is no edge (v, u) in both graphs, then it is possible to perform an operation with this pair of vertices, after which the problem reduces to the previous case.

If such an edge is not found, then all connected components of the first graph are cliques, and the second graph contains all edges that are not in the first graph.

Now, let’s select any connected component of the first graph, the size of which is at least 2 (if there is no such component, then perform an operation with any pair of vertices, after which such a connected component will appear). In this connected component, select arbitrary vertices a and b . In addition, from all other connected components of the first graph, select one vertex. Let’s call them v_1, v_2, \dots, v_k .

Notice that if the first graph contains at least three connected components or each connected component has a size of at least 2, then the following sequence of operations will work: $(a, v_1), (b, v_2), (b, v_3), \dots, (b, v_k)$.

If this is not the case, then the first graph contains an isolated vertex c , while all other vertices form a clique. In this case, the second graph is a star. In this case, if $n = 3$, there is no answer, otherwise, any pair of vertices a and b different from c can be chosen, and two operations (a, b) and (a, c) can be performed.

This allows solving the problem in $\mathcal{O}(n + m_1 + m_2)$ or $\mathcal{O}((n + m_1 + m_2) \log(m_1 + m_2))$ depending on the implementation.