Strong Connectivity Strikes Back

Subgroup 1

For each subset of edges, we will check if the Strongly Connected Components (SCCs) change when the edges in it are reversed. A good subset does not contain subsets for which the SCCs do not change. For each subset, we will check if it is good by enumerating all its subsets. Using Tarjan's or Kosaraju's algorithm to find SCCs, we obtain a solution in $O(3^m + 2^m \cdot m)$.

Subgroup 2

For each set, we need to check if it contains one of the "bad"subsets. We will use dynamic programming over subsets:

$$\text{is\_good}[S] = \neg\text{can\_reverse}[S] \wedge \bigwedge_{e \in S} \text{is\_good}[S \setminus \{e\}]$$

In total, we obtain a solution in $O(2^m \cdot m)$.

Subgroups 3, 4 (Acyclic Graph)

Consider some edge $e = (u, v)$. If there is no other path from u to v, e can be reversed, and the graph will remain acyclic. Thus, e cannot be in a good set.

Now suppose there is another path from u to v. Then we can restore the direction of e from the directions of the edges on this path, meaning the direction on e can be erased. The set of all such edges e is good. This set is the only maximal one.

To check for the existence of a path that does not go through an edge, we remove the edge from the graph and perform a breadth-first search. In total, we obtain a solution in $O(m^2)$.

Subgroup 5 (Hamiltonian Cycle)

We will call all edges that do not lie on the Hamiltonian cycle *direct*. Since reversing them does not violate the strong connectivity of the graph, they cannot be in a good set.

Direct edges cover certain segments of the cycle. Consider the intersection of several such segments (possibly disconnected, as the segments are cyclic). The claim is that all edges of the cycle that belong to this intersection can be reversed, and the graph will remain strongly connected.

Thus, at least one edge of the cycle in the intersection of segments must not lie in a good set. Note that if we can take another segment into our intersection and it decreases, we will obtain a stronger condition. Moreover, the "minimal"intersections that cannot be reduced further do not intersect.

We will select one edge from each minimal intersection and take the set of all edges except the direct ones and the selected ones. Why will it be good? Since the intersections are minimal, the edges in the graph form chains between vertices of degree 2. Since there should be no sinks and sources in the final graph, the directions on all edges of the chain are restored unambiguously.

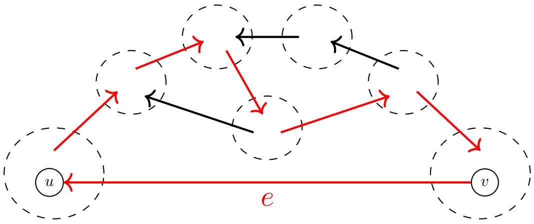
The resulting set will be a maximal good one. The number of ways is the product of the sizes of the minimal intersections.

Subgroup 6 (Strongly Connected Graph)

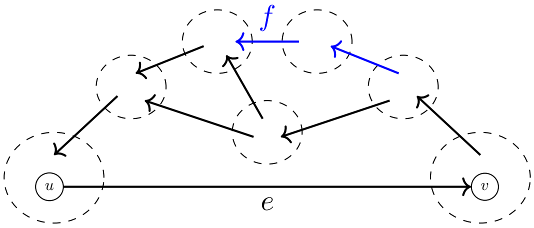
We generalize the idea of "minimal"sets that should not fully lie in a good one to the general case.

Consider an edge $e = (u, v)$. Again, if removing it keeps the graph strongly connected, e cannot be in a good set. Otherwise, reversing e would split the graph into several SCCs. We will denote the set of edges belonging to the condensation as $S(e)$ ($e \in S(e)$).

Suppose we have reversed e and want to reverse some subset of edges so that the graph remains strongly connected. We must reverse some subset $S(e)$ to create a path from u to v.



Note that if $f \in S(e)$, then $S(f) \subseteq S(e)$. If $S(f) = S(e)$ for all $f \in S(e)$, then the edges in $S(e)$ form a cycle in the condensation. We will call such $S(e)$ *necessary*—this is equivalent to the "minimal"set from the previous solution.



Consider some set of edges that can be reversed while keeping the graph strongly connected. Let e be an edge from this set with the minimal $S(e)$. Then $S(e)$ is necessary. Indeed, suppose it is not, then there exists an edge $f \in S(e)$ such that there is a path from v to u in the condensation graph that does not go through f. Then $S(f) \subset S(e)$; a contradiction.

Since $S(e)$ forms a cycle, $S(e)$ lies entirely within the considered set. Therefore, any set of edges that can be reversed without violating strong connectivity contains some necessary set.

Moreover, we can reverse $S(e)$ itself to keep the graph strongly connected. Thus, A is good $\iff$ A does not contain any necessary $S(e)$.

To achieve this, we need to not take one edge from each necessary set into A. Since they do not intersect, this can be done in any way. The maximal good set consists of all edges except those that do not affect strong connectivity and those selected from necessary sets. The number of ways is the product of the sizes of the necessary sets.

Complete Solution

We will solve the problem separately for each SCC and for the condensation graph. In the condensation graph, there may be several edges connecting two SCCs; we must not take any one of them into the good set. We will multiply the answer by their count.

The final solution works in $O(m^2)$.

Partial Score Solution

From the complete solution, an interesting fact follows: any good set lies within some maximal good set. This allows us to write the following greedy solution:

Suppose we have already chosen some good set. Now we want to understand if we can add another edge to it. Under what condition can we not do this? When it is possible to reverse our edge and some other subset of already erased edges while maintaining strong connectivity.

As mentioned above, if after reversing our edge (u, v) there is no longer a path from u to v, we need to reverse some edges from the already taken set to create this path. We can check if this is possible by searching for a path from u to v in the graph where we have added the reversed edges for each erased edge.

If such a path does not exist, then the edge (u, v) can be erased. Why, if such a path exists, can we not do this? It is claimed that after reversing the edge (u, v) and the edges on the found path, the graph will remain strongly connected. Indeed, we obtained a cycle from this path and the reversed edge (v, u). For all edges that we reversed on this path, reachability between the ends of the edge remained, as all vertices on the cycle are reachable from each other.

By considering all edges of the graph in any order, we will construct some maximal good set.